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INVARIANTS OF CURVES AND SURFACES OF THE SECOND DEGREE BY THE GROUP OF MOTIONS AND THE GROUP OF SIMILITUDE.

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One of the most important and interesting problems of Lie's Theory of Continuous Groups is the determination of functions and systems of equations that are left invariant by a given group of transformations. The solution of the problem to determine when, by change of variables and parameters, a given group can be transformed into another given group, reduces itself to that of finding the invariant systems of equations by a certain group, and, on the other hand, many problems in geometry and the theory of differential equations depend upon the determination of invariant configurations.*

Given, in particular, a continuous group of r parameters and n variables that is generated by the r independent infinitesimal point-transformations:

$$X_1 f,\ X_2 f,\ \ldots,\ X_r f;$$

where

$$X_i f = \xi_{i1}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_1} + \xi_{i2}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_2} + \ldots + \xi_{in}(x_1, \ldots, x_n)\frac{\partial f}{\partial x_n}.$$
$$i = 1, 2, \ldots, r.$$

The finite forms of the groups are

$$x_i' = x_i + \sum_1^r e_k X_k x_i + \frac{1}{1\,.\,2}\sum_1^r \sum_1^r e_k e_l X_k X_l x_i + \ldots, \quad i = 1, 2, \ldots, r$$

and any analytic function $f(x_1, \ldots, x_n)$ is transformed into the function

$$f'(x_1, \ldots, x_n) = f + \sum_1^r e_k X_k f + \frac{1}{1\,.\,2}\sum_1^r \sum_1^r e_k e_l X_k X_l f + \ldots, \quad i = 1, 2, \ldots, r$$

where, in both series, the e_k, e_l, $\ldots$, are constants.

Then if a function $I(x_1, \ldots, x_n)$ is to be an invariant function by the above r-parameter group there exists an identity of the form:

$$I(x_1, \ldots, x_n) + \sum_1^r e_k X_k I(x_1, \ldots, x_n) + \frac{1}{1\,.\,2}\sum_1^r \sum_1^r e_k e_l X_k X_l I(x_1, \ldots, x_n) + \ldots = I(x_1, \ldots, x_n) \quad (1)$$

that must be true for all values of the constants $e_1, \ldots, e_r$.

*See Sophus Lie—Vorlesungen über continuierliche Gruppen, bearbeitet von Dr. Scheffers, Leipzig, 1893, pp. 404 et seq.

Hence

$$\sum_1^r e_k X_k I = 0 ,$$

or

$$X_k I = 0 , \qquad k = 1, \ldots, r \quad (2)$$

consequently,*

$$X_k X_l I = 0 , \ldots ,$$

that is, I is an invariant function by the continuous group $X_1 f, X_2 f, \ldots, X_r f$ when the relations (2) are satisfied, and then only.

The equations (2) form a complete system of homogeneous linear partial differential equations of the first order, since

$$X_i (X_k f) - X_k (X_i f) = \sum_1^r c_{iks} X_s f .\dagger \quad (3)$$

Accordingly the determination of the invariant function $I(x_1, \ldots, x_n)$ depends upon the integration of the complete system

$$X_1 I = 0 , \; X_2 I = 0 , \; \ldots , \; X_r I = 0 . \quad (4)$$

This system contains r equations and n variables, and consequently has at least $n - r$ solutions. The problem of discovering the invariant functions by a group of infinitesimal point-transformations is thus referred by Lie to the integration of a complete system of homogeneous linear partial differential equations of the first order. Hence the number of invariant functions of n variables by a group of r parameters is not less than $n - r$. In particular cases the number of invariant functions may be greater than $n - r$ in consequence of the vanishing of certain determinants of the matrix formed by the coefficients of the $\frac{\partial f}{\partial x_j}$ in the complete system (4). The maximum number is obviously $n - 1$.

The solution above is independent of any special property of the infinitesimal point-transformations by which the continuous group considered is generated. It holds true, then, for a group of projective infinitesimal transformations. By an infinitesimal point-transformation, point is transformed into point, curve into curve, and surface into surface; the projective trans-

* This conclusion is also clear geometrically when we interpret $I(x_1, \ldots, x_n) = constant$ as a surface in space of n-dimensions. The equations (2) then express that the surfaces $I(x_1, \ldots, x_n) = a$, are generated by the path-curves of every infinitesimal point transformation $\sum e_k X_k f$ of the given group. Hence when $X_k = 0$, then also $X_k X_l = 0$, etc.

† In general the equations (2) form a complete system by virtue of the conditions (3) both when the c_{iks} are constants and when some or all are functions of the variables. But since, in the case in hand, the $X_i f$ form a continuous group, the first principles of Lie's theory demand that here the c_{iks} be constants.

formation possesses these properties together with the additional characteristic that the degree of the locus is also preserved by the transformation, that is, a surface of the mth degree goes over into a surface of the mth degree, and a curve of the nth degree into a curve of the nth degree.

Hence when the infinitesimal point-transformations $X_k f$ are projective the above method may be applied to determine the invariants by any projective group of 1° a surface of the mth degree; 2° a curve of the nth degree; 3° a system of points, of general or restricted position; 4° any system consisting of a finite number of points, curves, and surfaces.*

The values of the increments δx_i given to the variables x_i by the transformations are expressed explicitly in the transformations themselves. The increments $\delta_r a_j$ of the parameters a_j which enter into the equations of condition, namely, the equations of the figures that make up the geometric configuration whose invariants are to be determined, can be found by means of these equations. The $\delta_r a_j$ are obviously functions of the a_j alone.

It is proposed here to find the invariants† of a conic 1° by the group of motions, 2° by the group of similitude, in the plane; and to solve the corresponding problems in ordinary space, namely those of determining the invariants of a conicoid 1° by the group of motions, 2° by the group of similitude, in space of three dimensions.

I.

Considering the problem in the plane, let

$$U = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{5}$$

be the equation of the conic. Let

$$I(a, h, b, g, f, c)$$

be an invariant function of the coefficients of $U = 0$ by any projective group of infinitesimal transformations in the xy-plane. Then, since U is homogeneous in the parameters, I must be homogeneous and of the zero order, hence by Euler's theorem, we have

$$a\frac{\partial I}{\partial a} + h\frac{\partial I}{\partial h} + b\frac{\partial I}{\partial b} + g\frac{\partial I}{\partial g} + f\frac{\partial I}{\partial f} + c\frac{\partial I}{\partial c} = 0\,. \tag{6}$$

* Here the particular projective group considered is subject to but one limitation, namely: that it be a projective transformation in a space of dimensions not lower than the space of highest dimensions represented by any individual of the configuration whose invariant by the group is sought. For example, the method would not be applied to finding the invariants by the projective group of the plane of a system one of whose numbers is a sphere.

† The term "invariant" is here used in the sense of an "absolute invariant:" the determinant of substitution of Cayley's theory of invariants is here equal to unity. Lie has drawn a sharp distinction between invariant expressions and invariant equations. Vid. loc. cit. pp. 716 et seq.

Also since I is invariant, the variation of I must be zero, hence the equation

$$\delta I = \frac{\partial I}{\partial a}\delta a + \frac{\partial I}{\partial h}\delta h + \frac{\partial I}{\partial b}\delta b + \frac{\partial I}{\partial g}\delta g + \frac{\partial I}{\partial f}\delta f + \frac{\partial I}{\partial c}\delta c = 0 . \qquad (7)$$

The group of motions in the plane is a subgroup of the general projective group and contains the three infinitesimal point-transformations :

$$\boxed{p \;\; q \;\; yp - xq}\ * \qquad (8)$$

The group of similitude in the plane, also projective, is a four-parameter group and consists of the four infinitesimal point-transformations :

$$p \;\; q \;\; yp - xq \;\; xp - yq \qquad (9)$$

where $p = \dfrac{\partial f}{\partial x}$, $q = \dfrac{\partial f}{\partial y}$.

Operating on $U = 0$ with the several infinitesimal transformations of these groups, we obtain the following results :

$$pU\dagger = 2ax + 2hy + 2g + x^2\delta_1 a + 2xy\delta_1 h + y^2\delta_1 b + 2x\delta_1 g + 2y\delta_1 f + \delta_1 c = 0 , \qquad (10)$$

$$qU = 2hx + 2by + 2f + x^2\delta_2 a + 2xy\delta_2 h + y^2\delta_2 h + 2x\delta_2 g + 2y\delta_2 f + \delta_2 c = 0 , \qquad (11)$$

$$(yp - xq)\,U = 2axy + 2hy^2 + 2gy - 2hx^2 - 2bxy - 2fx + x^2\delta_3 a + 2xy\delta_3 h + y^2\delta_3 b + 2x\delta_3 g + 2y\delta_3 f + \delta_3 c = 0 , \qquad (12)$$

$$(xp + yq)\,U = 2ax^2 + 2hxy + 2gx + 2hxy + 2by^2 + 2fy + x^2\delta_4 a + 2xy\delta_4 h + y^2\delta_4 b + 2x\delta_4 g + 2y\delta_4 f + \delta_4 c = 0 . \qquad (13)$$

These equations exist by virtue of the equation $U = 0$; hence comparing them successively with $U = 0$, we have the following systems, respectively :

$$\frac{\delta_1 a}{a} = \frac{\delta_1 h}{h} = \frac{\delta_1 b}{b} = \frac{\delta_1 g + a}{g} = \frac{\delta_1 f + h}{f} = \frac{\delta_1 c + 2g}{c} = \rho_1 ; \qquad (14)$$

$$\frac{\delta_2 a}{a} = \frac{\delta_2 h}{h} = \frac{\delta_2 b}{b} = \frac{\delta_2 g + h}{g} = \frac{\delta_2 f + b}{f} = \frac{\delta_2 c + 2f}{c} = \rho_2 ; \qquad (15)$$

* Lie uses this notation to indicate that the infinitesimal transformations enclosed form a continuous group.

† The notation $pU, qU, \ldots$, is used for convenience only, in order to enable the reader to identify the results of the operations. It is not strictly exact, since here the $X_i f$ include also arbitrary operators giving rise to the terms in δx_j.

$$\frac{\delta_3 a - 2h}{a} = \frac{\delta_3 h + a - b}{h} = \frac{\delta_3 b + 2h}{b} = \frac{\delta_3 g - f}{g} = \frac{\delta_3 f + g}{f} = \frac{\delta_3 c}{c} = \rho_3; \quad (16)$$

$$\frac{\delta_4 a + 2h}{a} = \frac{\delta_4 h + 2h}{h} = \frac{\delta_4 b + 2b}{b} = \frac{\delta_4 g + g}{g} = \frac{\delta_4 f + f}{f} = \frac{\delta_4 c}{c} = \rho_4. \quad (17)$$

The solution of these systems of equations for the variations of the parameters gives :

$$\delta_1 a = \rho_1 a \quad , \delta_1 h = \rho_1 h \quad , \delta_1 b = \rho_1 b \quad , \delta_1 g = \rho_1 g - a,$$
$$\delta_1 f = \rho_1 f - h, \ \delta_1 c = \rho_1 c - 2g; \quad (18)$$

$$\delta_2 a = \rho_2 a \quad , \delta_2 h = \rho_2 h \quad , \delta_2 b = \rho_2 b \quad , \delta_2 g = \rho_2 g - h,$$
$$\delta_2 f = \rho_2 f - b, \ \delta_2 c = \rho_2 c - 2f; \quad (19)$$

$$\delta_3 a = \rho_3 a + 2h, \ \delta_3 h = \rho_3 h - (a - b), \ \delta_3 b = \rho_3 b - 2h, \ \delta_3 g = \rho_3 g + f,$$
$$\delta_3 f = \rho_3 f - g, \ \delta_3 c = \rho_3 c; \quad (20)$$

$$\delta_4 a = \rho_4 a - 2a, \ \delta_4 h = \rho_4 h - 2h \quad , \delta_4 b = \rho_4 b - 2b, \ \delta_4 g = \rho_4 g - g,$$
$$\delta_4 f = \rho_4 f - f, \ \delta_4 c = \rho_4 c.^*. \quad (21)$$

These systems of values must satisfy the equation (7), respectively. They contain the four indeterminate multipliers ρ_1, ρ_2, ρ_3, ρ_4, but these will be found to cause no inconvenience, since by virtue of the partial differential equation (6) they disappear from the results of the substitutions in equation (7) of the above values of the variations of the parameters ; in the results ρ appears only as the coefficient of the left hand member of equation (6). Making the substitutions successively in the above order we have respectively :

$$a\frac{\partial I}{\partial g} + h\frac{\partial I}{\partial f} + 2g\frac{\partial I}{\partial c} = 0, \quad (22)$$

$$h\frac{\partial I}{\partial g} + b\frac{\partial I}{\partial f} + 2f\frac{\partial I}{\partial c} = 0, \quad (23)$$

$$2h\left[\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b}\right] - (a - b)\frac{\partial I}{\partial h} + f\frac{\partial I}{\partial g} - g\frac{\partial I}{\partial f} = 0, \quad (24)$$

$$2a\frac{\partial I}{\partial a} + 2h\frac{\partial I}{\partial h} + 2b\frac{\partial I}{\partial b} + g\frac{\partial I}{\partial g} + f\frac{\partial I}{\partial f} = 0. \quad (25)$$

* **These values of the variations of the parameters might also have been found in this manner : Form the total variation of $U = 0$, viz. $\delta_r U = 0$ where δ_r operates on both variables and parameters, substitute in the resulting equations the values of $\delta_r x$ and $\delta_r y$ as expressed by the infinitesimal transformations considered ; compare the new equations respectively with $U = 0$, and we obtain as before the system (14) . . . (17).**

1. Then to determine the invariants of the conic $U = 0$ by the group of motions we have the following complete system of homogeneous linear partial differential equations of the first order :

$$M_2 \begin{cases} a\frac{\partial I}{\partial g} + h\frac{\partial I}{\partial f} + 2g\frac{\partial I}{\partial c} = 0, & (22) \\ h\frac{\partial I}{\partial g} + b\frac{\partial I}{\partial f} + 2f\frac{\partial I}{\partial c} = 0, & (23) \\ 2h\frac{\partial I}{\partial a} - (a-b)\frac{\partial I}{\partial h} - 2h\frac{\partial I}{\partial b} + f\frac{\partial I}{\partial g} - g\frac{\partial I}{\partial f} = 0, & (24) \\ a\frac{\partial I}{\partial a} + h\frac{\partial I}{\partial h} + b\frac{\partial I}{\partial b} + g\frac{\partial I}{\partial g} + f\frac{\partial I}{\partial f} + c\frac{\partial I}{\partial c} = 0. & (6) \end{cases}$$

This complete system, M_2, has six variables and four equations, and hence at least two independent solutions. In order to determine whether there exist other solutions we form the matrix of the coefficients of $\frac{\partial I}{\partial a}, \frac{\partial I}{\partial h}, \ldots,$

$$\begin{vmatrix} 0 & 0 & 0 & a & h & 2g \\ 0 & 0 & 0 & h & b & 2f \\ 2h & -2h & b-a & f & -g & 0 \\ a & h & b & g & f & c \end{vmatrix}$$

That the number of independent solutions of M_2 be greater than two it is necessary that all fourth order determinants of this matrix vanish identically. But this is easily seen not to be the case. Take, for example, the determinant formed by the first four columns on the right, in which c appears but once. The coefficient of c, namely the minor of c, is not zero, and hence, since the $a, h, b, \ldots,$ are independent, the determinant considered cannot vanish identically. Therefore the system M_2 has but two independent solutions.

Hence a conic possesses only two independent invariants by the group of motions. The integration of M_2 will now give these invariants.

For convenience put, as usual,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

$$A = bc - f^2,\ B = ca - g^2,\ C = ab - h^2,\ F = gh - af,\ G = hf - bg,\ H = fg - ch.$$

Note the following consequences of these identities:

$$\left.\begin{aligned} &aA + hH + gG = hH + bB + fF = \ldots = \Delta, \\ &hA + bH + fg = gA + fH + cG = \ldots = 0, \\ &BC - F^2 = a\Delta,\ CA - G^2 = b\Delta, \ldots. \end{aligned}\right\} \tag{26}$$

The equation (22) is equivalent to the following simultaneous system:

$$\frac{dg}{a} = \frac{df}{h} = \frac{dc}{2g};$$

whence at once are written the two integrals

$$B,\quad F.$$

Introducing these in equation (23) there results the partial differential equation

$$2F\frac{\partial I}{\partial B} - C\frac{\partial I}{\partial F} = 0$$

whose equivalent simultaneous system

$$\frac{dB}{2F} = -\frac{dF}{C}$$

gives the integral

$$BC - F^2,$$

or by the relations (26) simply

$$\Delta.$$

Substituting this as a variable in equation (24) and observing that the relations (26) render the coefficient of $\frac{\partial I}{\partial \Delta}$ zero, we have the partial differential equation:

$$2h\left[\frac{\partial I}{\partial a} - \frac{\partial I}{\partial b}\right] - (a - b)\frac{\partial I}{\partial h} = 0.$$

The equivalent simultaneous system,

$$-\frac{da}{2h} = \frac{db}{2h} = \frac{dh}{a - b},$$

leaves the integrals

$$a + b,\quad C.$$

We have then the three solutions of (22), (23) and (24), viz.,

$$a + b,\quad C,\quad \Delta.$$

Equation (6), however, demands that the common solutions of the system M_2 be homogeneous and of the zero order of homogeneity.

The forms

$$I_1 = \frac{(a+b)\,\Delta}{C^2}, \quad I_2 = \frac{\Delta^2}{C^3} \tag{27}$$

are easily seen to satisfy these conditions and the system. Therefore

$$\frac{(a+b)\,\Delta}{C^2} \text{ and } \frac{\Delta^2}{C^3}$$

are two independent invariants of the conic $U = 0$ by the group of motions, all other invariants of $U = 0$ by the group of motions are functions of these two.

Since I_1 and I_2 are absolute invariants it is easy to give a geometrical interpretation to them, and thereby we arrive at a simple equation for determining the magnitude of the axes of a conic in terms of the parameters of its general equation. For, taking the equation of the conic in the normal form $V = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = 0$, we have

$$a = \frac{1}{\alpha^2}, \; b = \frac{1}{\beta^2}, \; c = -1, \; f = g = h = 0, \; \Delta = -\frac{1}{\alpha^2\beta^2}, \; C = +\frac{1}{\alpha^2\beta^2},$$

$$a + b = \frac{1}{\alpha^2} + \frac{1}{\beta^2};$$

hence

$$I_1 = -(\alpha^2 + \beta^2), \quad I_2 = +\alpha^2\beta^2.$$

Accordingly the axes of the conic $V = 0$ are invariant by the group of motions, and the two invariants I_1 and I_2 express respectively the sum of the squares and the product of the squares of the semi-axes of $V = 0$. Consequently, since the expressions I_1 and I_2 are absolute invariants, they are the expressions, respectively, for the sum of the squares and the product of the squares of the semi-axes of the conic $U = 0$; and the axes of $U = 0$ are invariant by the group of motions.

Hence the squares of the semi-axes of the conic $U = 0$ are given by the quadratic equation

$$\lambda^2 + I_1\lambda + I_2 = 0,$$

or

$$C^3\lambda^2 + (a+b)\,C\Delta\lambda + \Delta^2 = 0.^* \tag{28}$$

A discussion of the roots of this equation would lead to an exhaustive classi-

* A similar but more complicated form is given without proof in the sixth edition of Salmon's Conic Sections, Miscellaneous Notes, p. 392.

fication of the curves of the second degree. Also the invariance of I_1 and I_2 constitutes the Congruence Criterium* of the curves of the second degree.

2. To determine the invariants of $U = 0$ by the group of similitude it is necessary to integrate the complete system :

$$S_2 \left\{ \begin{array}{l} a\dfrac{\partial I}{\partial g} + h\dfrac{\partial I}{\partial f'} + 2g\dfrac{\partial I}{\partial c} = 0\,, \qquad (22) \\ h\dfrac{\partial I}{\partial g} + b\dfrac{\partial I}{\partial f'} + 2f\dfrac{\partial I}{\partial c} = 0\,, \qquad (23) \\ 2h\dfrac{\partial I}{\partial a} - (a-b)\dfrac{\partial I}{\partial h} - 2h\dfrac{\partial I}{\partial b} + f'\dfrac{\partial I}{\partial g} - g\dfrac{\partial I}{\partial f'} = 0\,, \qquad (24) \\ 2a\dfrac{\partial I}{\partial a} + 2h\dfrac{\partial I}{\partial h} + 2b\dfrac{\partial I}{\partial b} + g\dfrac{\partial I}{\partial g} + f'\dfrac{\partial I}{\partial f'} = 0\,, \qquad (25) \\ a\dfrac{\partial I}{\partial a} + h\dfrac{\partial I}{\partial h} + b\dfrac{\partial I}{\partial b} + g\dfrac{\partial I}{\partial g} + f'\dfrac{\partial I}{\partial f'} + c\dfrac{\partial I}{\partial c} = 0\,. \qquad (6) \end{array} \right.$$

Here are five equations and six variables ; accordingly, the system has at least one solution. To determine if there are others it is only necessary to form the matrix as in the preceding problem, when it will be seen that all fifth order determinants of the matrix do not vanish identically ; consequently there are no additional solutions and the conic $U = 0$ has but one invariant by the group of similitude. This invariant may now be found by the integration of the system S_2, which is accomplished without difficulty when we avail ourselves of the integrals found in the preceding problem. The solution then is readily seen to be

$$I_3 = \frac{(a+b)^2}{C}\,; \qquad (29)$$

and the geometrical interpretation of I_3 is easily found to be

$$I_3 = \frac{(a+b)^2}{C} = \frac{I_1^2}{I_2} = (\alpha^2 + \beta^2)\left[\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right],\dagger$$

where α and β are the semi-axes of $U = 0$. Hence the ratio of the axes of the conic is invariant by the group of similitude. The invariance of I_3 then includes the whole theory of the similar curves of the second degree. As a corollary the eccentricity depends only on the coefficients of the terms of the second degree since $I_3 = \dfrac{(a+b)^2}{ab - h^2}$.

* Vid. loc. cit., pp. 673 et seq.

† It is to be observed that I_1 and I_2 are not invariants in these equations.

II.

Consider now the corresponding problems in space. Let

$$T = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}xz + 2a_{12}xy + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0\,, \qquad (30)$$

be the equation of the conicoid. Let

$$J(a_{ij})$$

be an invariant function of the parameters of $T = 0$ by a projective group of infinitesimal transformations in ordinary space. T is homogeneous in these parameters, hence J is a homogeneous function and of the zero order; accordingly Euler's theorem gives

$$\Sigma a_{ij}\frac{\partial J}{\partial a_{ij}} = 0\,. \quad (a_{ij} = a_{ji},\ i, j = 1, 2, 3, 4) \qquad (31)$$

Also the invariance of J gives

$$\delta J \quad \Sigma \frac{\partial J}{\partial a_{ij}}\delta a_{ij} = 0\,. \quad (a_{ij} = a_{ji},\ i, j = 1, 2, 3, 4) \qquad (32)$$

The group of motions in space is a six-parameter projective group and consists of the infinitesimal transformations:

$$\boxed{p \;\; q \;\; r \;\; yp - xq \;\; zq - yr \;\; xr - zp} \qquad (33)$$

The group of similitude in space, also projective, contains the seven infinitesimal transformations:

$$\boxed{p \;\; q \;\; r \;\; yp - xq \;\; zq - yr \;\; xr - zp \;\; xp - yq + zr} \qquad (34)$$

where $p = \dfrac{\partial f}{\partial x}$, $q = \dfrac{\partial f}{\partial y}$, $r = \dfrac{\partial f}{\partial z}$.

Operating on $T = 0$ with the several infinitesimal transformations of these groups, we have

$$\begin{aligned} pT^* \quad & \delta_1 a_{11}x^2 + \delta_1 a_{22}y^2 + \delta_1 a_{33}z^2 + 2\delta_1 a_{23}yz + 2\delta_1 a_{13}xz + 2\delta_1 a_{12}xy \\ & + 2(a_{11} + \delta_1 a_{14})x + 2(a_{12} + \delta_1 a_{24})y + 2(a_{13} + \delta_1 a_{34})z \\ & + \delta_1 a_{44} + 2a_{14} = 0\,; \end{aligned} \qquad (35)$$

* It is to be noted that the notation $pT, \ldots$ is not rigorously exact and that it is used only for purposes of identification.

$$\begin{aligned} qT = {} & \delta_2 a_{11}x^2 + \delta_2 a_{22}y^2 + \delta_2 a_{33}z^2 + 2\delta_2 a_{23}yz + 2\delta_2 a_{13}xz + 2\delta_2 a_{12}xy \\ & + 2(a_{12} + \delta_2 a_{14})x + 2(a_{22} + \delta_2 a_{24})y + 2(a_{23} + \delta_2 a_{44})z \\ & + \delta_2 a_{44} + 2a_{24} = 0\,; \end{aligned} \tag{36}$$

$$\begin{aligned} rT = {} & \delta_3 a_{11}x^2 + \delta_3 a_{22}y^2 + \delta_3 a_{33}z^2 + 2\delta_3 a_{23}yz + 2\delta_3 a_{13}xz + 2\delta_3 a_{12}xy \\ & + 2(a_{13} + \delta_3 a_{14})x + 2(a_{23} + \delta_3 a_{24})y + 2(a_{33} + \delta_3 a_{34})z \\ & + \delta_3 a_{44} + 2a_{34} = 0\,; \end{aligned} \tag{37}$$

$$\begin{aligned} (yp - xq)\,T = {} & (\delta_4 a_{11} - 2a_{12})x^2 + (\delta_4 a_{22} + 2a_{12})y^2 + \delta_4 a_{33}z^2 + 2(\delta_4 a_{23} + a_{13})yz \\ & + 2(\delta_4 a_{13} - a_{23})xz + 2(\delta_4 a_{12} + a_{11} - a_{22})xy + 2(\delta_4 a_{14} - a_{24})x \\ & + 2(\delta_4 a_{24} + a_{14})y + 2\delta_4 a_{34}z + \delta_4 a_{44} = 0\,; \end{aligned} \tag{38}$$

$$\begin{aligned} (zq - xr)\,T = {} & \delta_5 a_{11}x^2 + (\delta_5 a_{22} - 2a_{23})y^2 + (\delta_5 a_{33} + 2a_{23})z^2 \\ & + 2(\delta_5 a_{23} + a_{22} - a_{33})yz + 2(\delta_5 a_{13} + a_{23})xz + 2(\delta_5 a_{23} - a_{13})xy \\ & + 2\delta_5 a_{14}x + 2(\delta_5 a_{24} - a_{34})y + 2(\delta_5 a_{34} + a_{24})z + \delta_5 a_{44} = 0\,; \end{aligned} \tag{39}$$

$$\begin{aligned} (zr - zp)\,T = {} & (\delta_6 a_{11} + 2a_{13})x^2 + \delta_6 a_{22}y^2 + (\delta_6 a_{33} - 2a_{13})z^2 + 2(\delta_6 a_{23} - a_{12})yz \\ & + 2(\delta_6 a_{13} + a_{33} - a_{11})xz + 2(\delta_6 a_{12} + a_{23})xy + 2(\delta_6 a_{14} + a_{34})x \\ & + 2\delta_6 a_{24}y + 2(\delta_6 a_{34} - a_{14})z + \delta_6 a_{44} = 0\,; \end{aligned} \tag{40}$$

$$\begin{aligned} (xp + yq + zr)\,T = {} & (\delta_7 a_{11} + 2a_{11})x^2 + (\delta_7 a_{22} + 2a_{22})y^2 + (\delta_7 a_{33} + 2a_{33})z^2 \\ & + 2(\delta_7 a_{23} + 2a_{23})yz + 2(\delta_7 a_{13} + 2a_{13})xz + 2(\delta_7 a_{12} + 2a_{12})xy \\ & + 2(\delta_7 a_{14} + a_{14})x + 2(\delta_7 a_{24} + a_{24})y + 2(\delta_7 a_{34} + a_{34})z \\ & + \delta_7 a_{44} = 0\,. \end{aligned} \tag{41}$$

The above equations exist by virtue of the equation $T = 0$; hence comparing them successively with $T = 0$, we have respectively:

$$\begin{aligned} \frac{\delta_1 a_{11}}{a_{11}} = \frac{\delta_1 a_{22}}{a_{22}} = \frac{\delta_1 a_{33}}{a_{33}} = \frac{\delta_1 a_{23}}{a_{23}} = \frac{\delta_1 a_{13}}{a_{13}} = \frac{\delta_1 a_{12}}{a_{12}} = \frac{\delta_1 a_{14} + a_{11}}{a_{14}} = \frac{\delta_1 a_{24} + a_{12}}{a_{24}} \\ = \frac{\delta_1 a_{34} + a_{13}}{a_{34}} = \frac{\delta_1 a_{44} + 2a_{14}}{a_{44}} = \rho_1\,; \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\delta_2 a_{11}}{a_{11}} = \frac{\delta_2 a_{22}}{a_{22}} = \frac{\delta_2 a_{33}}{a_{33}} = \frac{\delta_2 a_{23}}{a_{23}} = \frac{\delta_2 a_{13}}{a_{13}} = \frac{\delta_2 a_{12}}{a_{12}} = \frac{\delta_2 a_{14} + a_{12}}{a_{14}} = \frac{\delta_2 a_{24} + a_{22}}{a_{24}} \\ = \frac{\delta_2 a_{34} + a_{23}}{a_{34}} = \frac{\delta_2 a_{44} + 2a_{24}}{a_{44}} = \rho_2\,; \end{aligned} \tag{43}$$

$$\begin{aligned} \frac{\delta_3 a_{11}}{a_{11}} = \frac{\delta_3 a_{22}}{a_{22}} = \frac{\delta_3 a_{33}}{a_{33}} = \frac{\delta_3 a_{23}}{a_{23}} = \frac{\delta_3 a_{13}}{a_{13}} = \frac{\delta_3 a_{12}}{a_{12}} = \frac{\delta_3 a_{14} + a_{13}}{a_{14}} = \frac{\delta_3 a_{24} + a_{23}}{a_{24}} \\ = \frac{\delta_3 a_{34} + a_{33}}{a_{34}} = \frac{\delta_3 a_{44} + 2a_{34}}{a_{44}} = \rho_3\,; \end{aligned} \tag{44}$$

$$\frac{\delta_4 a_{11} - 2a_{12}}{a_{11}} = \frac{\delta_4 a_{22} + 2a_{12}}{a_{22}} = \frac{\delta_4 a_{33}}{a_{33}} = \frac{\delta_4 a_{23} + a_{13}}{a_{23}} = \frac{\delta_4 a_{13} - a_{23}}{a_{13}} = \frac{\delta_4 a_{12} + a_{11} - a_{22}}{a_{12}}$$
$$= \frac{\delta_4 a_{14} - a_{24}}{a_{14}} = \frac{\delta_4 a_{24} + a_{14}}{a_{24}} = \frac{\delta_4 a_{34}}{a_{34}} = \frac{\delta_4 a_{44}}{a_{44}} = \rho_4 ; \quad (45)$$

$$\frac{\delta_5 a_{11}}{a_{11}} = \frac{\delta_5 a_{22} - a_{23}}{a_{22}} = \frac{\delta_5 a_{33} + 2a_{23}}{a_{33}} = \frac{\delta_5 a_{23} + a_{22} - a_{33}}{a_{23}} = \frac{\delta_5 a_{13} + a_{12}}{a_{13}} = \frac{\delta_5 a_{12} - a_{13}}{a_{12}}$$
$$= \frac{\delta_5 a_{14}}{a_{14}} = \frac{\delta_5 a_{24} - a_{34}}{a_{24}} = \frac{\delta_5 a_{34} + a_{24}}{a_{34}} = \frac{\delta_5 a_{44}}{a_{44}} = \rho_5 ; \quad (46)$$

$$\frac{\delta_6 a_{11} + 2a_{13}}{a_{11}} = \frac{\delta_6 a_{22}}{a_{22}} = \frac{\delta_6 a_{33} - 2a_{13}}{a_{33}} = \frac{\delta_6 a_{23} - a_{12}}{a_{23}} = \frac{\delta_6 a_{13} + a_{33} - a_{11}}{a_{13}} = \frac{\delta_6 a_{12} + a_{23}}{a_{12}}$$
$$= \frac{\delta_6 a_{14} + a_{34}}{a_{14}} = \frac{\delta_6 a_{24}}{a_{24}} = \frac{\delta_6 a_{34} - a_{14}}{a_{34}} = \frac{\delta_6 a_{44}}{a_{44}} = \rho_6 ; \quad (47)$$

$$\frac{\delta_7 a_{11} + 2a_{11}}{a_{11}} = \frac{\delta_7 a_{22} + 2a_{22}}{a_{22}} = \frac{\delta_7 a_{33} + 2a_{33}}{a_{33}} = \frac{\delta_7 a_{23} + 2a_{23}}{a_{23}} = \frac{\delta_7 a_{13} + 2a_{13}}{a_{13}}$$
$$= \frac{\delta_7 a_{12} + 2a_{12}}{a_{12}} = \frac{\delta_7 a_{14} + a_{14}}{a_{14}} = \frac{\delta_7 a_{24} + a_{24}}{a_{24}} = \frac{\delta_7 a_{34} + a_{24}}{a_{34}} = \frac{\delta_7 a_{44}}{a_{44}} = \rho_7 . \quad (48)$$

The values of the $\delta_r a_{ij}$, now to be found by the solutions of these systems of equations, must satisfy the equation (32). By virtue of equation (31) the indeterminate multipliers $\rho_1, \ldots, \rho_7$ disappear after the substitution of the $\delta_r a_{ij}$ in equation (32). Hence the solution of equations (42) to (48) with regard to the $\delta_r a_{ij}$ and the successive substitution of the systems of values of $\delta_r a_{ij}$ in the equation (32) leaves the following homogeneous linear partial differential equations of the first order, respectively:

$$a_{11}\frac{\partial J}{\partial a_{14}} + a_{12}\frac{\partial J}{\partial a_{24}} + a_{13}\frac{\partial J}{\partial a_{34}} + 2a_{14}\frac{\partial J}{\partial a_{44}} = 0 , \quad (49)$$

$$a_{12}\frac{\partial J}{\partial a_{14}} + a_{22}\frac{\partial J}{\partial a_{24}} + a_{23}\frac{\partial J}{\partial a_{34}} + 2a_{24}\frac{\partial J}{\partial a_{44}} = 0 , \quad (50)$$

$$a_{13}\frac{\partial J}{\partial a_{14}} + a_{23}\frac{\partial J}{\partial a_{24}} + a_{33}\frac{\partial J}{\partial a_{34}} + 2a_{34}\frac{\partial J}{\partial a_{44}} = 0 , \quad (51)$$

$$2a_{12}\left[\frac{\partial J}{\partial a_{11}} - \frac{\partial J}{\partial a_{22}}\right]$$
$$- a_{13}\frac{\partial J}{\partial a_{13}} + a_{23}\frac{\partial J}{\partial a_{23}} - (a_{11} - a_{22})\frac{\partial J}{\partial a_{12}} + a_{24}\frac{\partial J}{\partial a_{14}} - a_{14}\frac{\partial J}{\partial a_{24}} = 0 , \quad (52)$$

$$2a_{23}\left[\frac{\partial J}{\partial a_{22}} - \frac{\partial J}{\partial a_{33}}\right]$$
$$- (a_{22} - a_{33})\frac{\partial J}{\partial a_{23}} - a_{12}\frac{\partial J}{\partial a_{13}} + a_{13}\frac{\partial J}{\partial a_{12}} + a_{34}\frac{\partial J}{\partial a_{24}} - a_{24}\frac{\partial J}{\partial a_{34}} = 0 , \quad (53)$$

$$2a_{13}\left\{\frac{\partial J}{\partial a_{33}} - \frac{\partial J}{\partial a_{11}}\right\}$$
$$+ a_{12}\frac{\partial J}{\partial a_{23}} - (a_{33} - a_{11}) - a_{23}\frac{\partial J}{\partial a_{12}} - a_{34}\frac{\partial J}{\partial a_{14}} + a_{14}\frac{\partial J}{\partial a_{34}} = 0\,, \quad (54)$$

$$2\left\{a_{11}\frac{\partial J}{\partial a_{11}} + a_{22}\frac{\partial J}{\partial a_{22}} + a_{33}\frac{\partial J}{\partial a_{33}} + a_{23}\frac{\partial J}{\partial a_{23}} + a_{13}\frac{\partial J}{\partial a_{13}} + a_{12}\frac{\partial J}{\partial a_{12}}\right\}$$
$$+ a_{14}\frac{\partial J}{\partial a_{14}} + a_{24}\frac{\partial J}{\partial a_{24}} + a_{34}\frac{\partial J}{\partial a_{34}} = 0\,. \quad (55)$$

3. Then to determine the invariants of the conicoid $T = 0$ by the group of motions in space we have the complete system, M_3, of homogeneous linear partial differential equations of the first order composed of the equations

(49) (50) (51) (52) (53) (54) (31).

This is a system of seven equations in ten variables and hence has at least three solutions. By forming the matrix whose elements are the coefficients of the $\frac{\partial J}{\partial a_{ij}}$ in the above system it is readily seen that all seventh order determinants of the matrix do not vanish identically, and that accordingly there are no more than three independent solutions. In this matrix, namely,

0	0	0	0	0	0	a_{11}	a_{12}	a_{13}	$2a_{14}$
0	0	0	0	0	0	a_{12}	a_{22}	a_{23}	$2a_{24}$
0	0	0	0	0	0	a_{13}	a_{23}	a_{33}	$2a_{34}$
$2a_{12}$	$-2a_{12}$	0	$-a_{13}$	a_{23}	$-(a_{11}-a_{22})$	a_{24}	$-a_{14}$	0	0
0	$2a_{23}$	$-2a_{23}$	$-(a_{22}-a_{33})$	$-a_{12}$	a_{13}	0	a_{34}	$-a_{24}$	0
$-2a_{13}$	0	$2a_{13}$	a_{12}	$-(a_{33}-a_{11})$	$-a_{23}$	$-a_{34}$	0	a_{14}	0
a_{11}	a_{22}	a_{33}	a_{23}	a_{13}	a_{12}	a_{14}	a_{24}	a_{34}	a_{44}

the term a_{44} occurs but once. Its coefficient in the determinant formed by the seven columns on the right is not zero; hence this determinant itself cannot be equal to zero, since the a_{ij} are independent of one another.

Hence the conicoid $T = 0$ has but three independent invariants by the group of motions. These may now be found by integrating the complete system M_3.

For convenience put

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix},$$

and A_{ii}, $2A_{ij}$ identically equal respectively to the minors of D corresponding to a_i^i, a_{ij}.

Then proceeding with the system M_3 in exactly the same manner in which the system M_2 of the corresponding problem in the plane was dealt with, we find that the first three equations (49), (50) and (51) have the solution

$$A_{44}.$$

Further that the equations (52), (53) and (54) have in addition to this the solutions

$$a_{11} + a_{22} + a_{33},\ a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{23}^2 - a_{13}^2 - a_{12}^2,\ D.$$

Finally the condition of homogeneity demanded by the equation (31) and the system are found to be satisfied by

$$J_1 = \frac{DP}{A_{44}^2}, \quad J_2 = \frac{D^2S}{A_{44}^3}, \quad J_3 = \frac{D^3}{A_{44}^4},$$

where $S = a_{11} + a_{22} + a_{33}$, and $P = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{23}^2 - a_{13}^2 - a_{12}^2$.

Analogous to the geometrical interpretation in the plane we find on comparison with the conicoid

$$W = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} - 1 = 0,$$

that

$$J_1 = -(\alpha^2 + \beta^2 + \gamma^2), \quad J_2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2, \quad J_3 = -\alpha^2\beta^2\gamma^2.$$

Consequently the axes of $T = 0$ are invariants by the group of motions, and the squares of the semi-axes are given by the cubic equation

$$\mu^3 + J_1\mu^2 + J_2\mu + J_3 = 0$$

or

$$A_{44}^4\mu^3 + A_{44}^2DP\mu^2 + A_{44}D^2S\mu + D^3 = 0. \tag{56}$$

The surfaces of the second degree may be classified by a discussion of the roots of this equation. It will also be observed that the Congruence Criterium of surfaces of the second degree is expressed by the invariance of J_1, J_2, and J_3.

4. To find the invariants of the conicoid $T = 0$ by the group of similitude in space (34), we have only to add the partial differential equation (55) to the system M_3. The new system S_3 consisting of the partial differential equations

$$(49)\ (50)\ (51)\ (52)\ (53)\ (54)\ (55)\ (31)$$

is a complete system of eight equations in ten variables. As in the preceding problem, a_{44} appears but once in the matrix, whereby it is easily seen that all eighth order determinants of the matrix do not vanish identically. The direct integration is simple when the solutions already found, namely, J_1, J_2, J_3 above, are made use of, and the two independent solutions of the system appear in the forms

$$J_4 = \frac{P^3}{A_{44}^2} = \frac{J_1^3}{J_3}, \quad J_5 \quad \frac{PS}{A_{44}} = \frac{J_1 J_2^*}{J_3}. \tag{57}$$

As in the plane problem these invariants determine the ratios of the axes of the conicoid. Hence the ratios of the axes of the conicoid $T = 0$ are invariant by the group of similitude in space.† The invariance of J_4 and J_5 expresses the condition that surfaces of the second degree be similar.

5. The method of this note allows of extension directly to curves whose equations take the form

$$\frac{x^{2n}}{\alpha^{2n}} + \frac{y^{2n}}{\beta^{2n}} = 1,$$

to the surfaces

$$\frac{x^{2n}}{\alpha^{2n}} + \frac{y^{2n}}{\beta^{2n}} + \frac{z^{2n}}{\gamma^{2n}} = 1,$$

and to the corresponding surfaces in spaces of higher dimensions, since these surfaces possess axes of symmetry at right angles. Obviously, as already remarked the method is capable of general application to curves and surfaces of any order, but the geometrical interpretation is not always reached with the same facility as in the cases above. Moreover a complete theory of the invariants 1° of a conic and 2° of systems of conics by the projective group in the plane may be developed by determining these invariants by the projective subgroups of one, two, three, and four parameters in the first case, and of parameters one, two, and so on to one less in number than the number of different constants entering the system in the second case. A corresponding theory for surfaces of the second degree is not without interest, but much more complicated. It is hoped to present some of these results in a subsequent note.

* The reader will observe that J_1, J_2, and J_3 are not invariants in these expressions.

† It is interesting to remark that the ratios of the axes of $T = 0$ are integrals of the complete system of partial differential equations S_3, and that the axes themselves are integrals of the system M_3. A similar remark applies to the corresponding problems in the plane.

CHRISTIANIA, 14 *April*, 1896.

SOLUTION OF A SYSTEM OF EQUATIONS OCCURRING IN DARBOUX'S *THÉORIE GÉNÉRALE DES SURFACES.**

By Dr. T. Craig, Baltimore, Md.

The equations are, to notation *près*,

$$x^2 + y^2 + z^2 = \text{const.}$$

(A) $$xx_1 + yy_1 + zz_1 = \text{const.}$$

$$xx_2 + yy_2 + zz_2 = \text{const.},$$

and we have also

$$x_1^2 + y_1^2 + z_1^2 = \text{const.}$$

$$x_2^2 + y_2^2 + z_2^2 = \text{const.}$$

It is required to find x, y, z from the first three of these. Darboux gives as the solution, without any indication of the process employed, the following:

$$x = A_1x_1 + A_2x_2 + A_3(y_1z_2 - y_2z_1),$$
$$y = A_1y_1 + A_2y_2 + A_3(z_1x_2 - z_2x_1),$$
$$z = A_1z_1 + A_2z_2 + A_3(x_1y_2 - x_2y_1).$$

A method for the solution of a general system of equations of the form (A) exists due to Bauer, but for the present case the following direct method, which is based on the most elementary geometrical considerations, seems to me preferable. The geometrical interpretation of the results and the notation is so obvious that it need not be referred to.

Write the equations in the form

$$x^2 + y^2 + z^2 = C^2,$$
$$x_1^2 + y_1^2 + z_1^2 = C_1^2, \tag{1}$$
$$x_2^2 + y_2^2 + z_2^2 = C_2^2,$$

$$xx_1 + yy_1 + zz_1 = CC_1\lambda_1,$$
$$xx_2 + yy_2 + zz_2 = CC_2\lambda_2; \tag{2}$$

where C, C_1, C_2 are arbitrary constants. Make now

$$x, y, z = C\alpha, C\beta, C\gamma,$$
$$x_1, y_1, z_1 = C_1\alpha_1, C_1\beta_1, C_1\gamma_1, \tag{3}$$
$$x_2, y_2, z_2 = C_2\alpha_2, C_2\beta_2, C_2\gamma_2.$$

* t. I, p. 21; the equations preceding (6).

The three equations which we have ultimately to solve are now

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= 1 \\ \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 &= \lambda_1 \\ \alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2 &= \lambda_2 \end{aligned} \tag{4}$$

with

$$\begin{aligned} \alpha_1^2 + \beta_1^2 + \gamma_1^2 &= 1 \\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 &= 1 . \end{aligned}$$

Define three new quantities x_3, y_3, z_3 by the equations

$$\begin{aligned} x_3^2 + y_3^2 + z_3^2 &= C_3^2 , \\ x_1x_3 + y_1y_3 + z_1z_3 &= 0 , \\ x_2x_3 + y_2y_3 + z_2z_3 &= 0 ; \end{aligned} \tag{5}$$

also write

$$xx_3 + yy_3 + zz_3 = CC_3\lambda_3'$$

Making

$$x_3, y_3, z_3 = C_3\alpha_3, C_3\beta_3, C_3\gamma_3$$

we have

$$\begin{aligned} \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1 , \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 &= 0 , \\ \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0 , \\ \alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3 &= \lambda_3' . \end{aligned} \tag{6}$$

The second and third of these give

$$\frac{\alpha_3}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\beta_3}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\gamma_3}{\alpha_1\beta_2 - \alpha_2\beta_1} . \tag{7}$$

Write

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = \cos\theta ; \tag{8}$$

square the terms in (7), add the numerators together and the denominators together, use the first of (6) and extract the square root of the result, this will be the common value of the ratios in (6), viz:

$$\frac{\alpha_3}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\beta_3}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\gamma_3}{\alpha_1\beta_2 - \alpha_2\beta_1} = \frac{1}{\sin\theta} , \tag{9}$$

or

$$\alpha_3 = \frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\sin\theta} , \quad \beta_3 = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\sin\theta} , \quad \gamma_3 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\sin\theta} . \tag{10}$$

These satisfy identically the first three of equations (6). Take now the last two of equations (4) and the last of (6); these are

$$\begin{aligned} \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 &= \lambda_1 , \\ \alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2 &= \lambda_2 , \\ \alpha\alpha_3 + \beta\beta_3 + \gamma\gamma_3 &= \lambda_3 . \end{aligned} \tag{11}$$

Substituting the above values of α_3, β_3, γ_3, these become

$$\begin{aligned} &\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = \lambda_3' \sin\theta = \lambda_3 , \\ &\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = \lambda_1 , \\ &\alpha\alpha_2 + \beta\beta_2 + \gamma\gamma_2 = \lambda_2 . \end{aligned} \tag{12}$$

Solving these we find

$$\begin{aligned} \alpha &= \frac{\lambda_3}{\sin^2\theta}(\beta_1\gamma_2 - \beta_2\gamma_1) + \frac{\lambda_2}{\sin^2\theta}(\alpha_2 - \alpha_1\cos\theta) + \frac{\lambda_1}{\sin^2\theta}(\alpha_1 - \alpha_2\cos\theta) , \\ \beta &= \frac{\lambda_3}{\sin^2\theta}(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \frac{\lambda_2}{\sin^2\theta}(\beta_2 - \beta_1\cos\theta) + \frac{\lambda_1}{\sin^2\theta}(\beta_1 - \beta_2\cos\theta) , \\ \gamma &= \frac{\lambda_3}{\sin^2\theta}(\alpha_1\beta_2 - \alpha_2\beta_1) + \frac{\lambda_2}{\sin^2\theta}(\gamma_2 - \gamma_1\cos\theta) + \frac{\lambda_1}{\sin^2\theta}(\gamma_1 - \gamma_2\cos\theta) . \end{aligned} \tag{13}$$

Write

$$\frac{\lambda_1}{\sin^2\theta} - \frac{\lambda_2\cos\theta}{\sin^2\theta} = l_1 , \quad \lambda_1 = l_1 + l_2\cos\theta ,$$

$$-\frac{\lambda_1\cos\theta}{\sin^2\theta} + \frac{\lambda_2}{\sin^2\theta} = l_2 , \quad \lambda_2 = l_2 + l_1\cos\theta .$$

Equations (13) now become

$$\begin{aligned} \alpha &= l_1\alpha_1 + l_2\alpha_2 + \frac{\lambda_3}{\sin^2\theta}(\beta_1\gamma_2 - \beta_2\gamma_1) , \\ \beta &= l_1\beta_1 + l_2\beta_2 + \frac{\lambda_3}{\sin^2\theta}(\gamma_1\alpha_2 - \gamma_2\alpha_1) , \\ \gamma &= l_1\gamma_1 + l_2\gamma_2 + \frac{\lambda_3}{\sin^2\theta}(\alpha_1\beta_2 - \alpha_2\beta_1) . \end{aligned} \tag{14}$$

To determine λ_3 (which with the auxiliaries α_3, β_3, γ_3 is not arbitrary) we use the first of (4), viz.,

$$\alpha^2 + \beta^2 + \gamma^2 = 1 . \tag{15}$$

This gives at once

$$\frac{\lambda_3}{\sin\theta} = \sqrt{1 - l_1^2 - l_2^2 - 2l_1 l_2 \cos\theta} \Big/ \sin\theta = \text{, say, } l_3 . \tag{16}$$

So we have now

$$\begin{aligned} \alpha &= l_1\alpha_1 + l_2\alpha_2 + l_3(\beta_1\gamma_2 - \beta_2\gamma_1), \\ \beta &= l_1\beta_1 + l_2\beta_2 + l_3(\gamma_1\alpha_2 - \gamma_2\alpha_1), \\ \gamma &= l_1\gamma_1 + l_2\gamma_2 + l_3(\alpha_1\beta_2 - \alpha_2\beta_1). \end{aligned} \tag{17}$$

These are of the required form ; to get back to x, y, z write

$$l_1, l_2, l_3 = C_1 m_1, C_2 m_2, C_1 C_2 m_3 ;$$

then multiply each of (17) through by the arbitrary constant C and again write

$$Cm_1, Cm_2, Cm_3 = A_1, A_2, A_3,$$

where A_1, A_2, A_3 are arbitrary constants, and we have finally

$$\begin{aligned} x &= A_1 x_1 + A_2 x_2 + A_3(y_1 z_2 - y_2 z_1), \\ y &= A_1 y_1 + A_2 y_2 + A_3(z_1 x_2 - z_2 x_1), \\ z &= A_1 z_1 + A_2 z_2 + A_3(x_1 y_2 - x_2 y_1), \end{aligned}$$

the required values. A number of modifications of the preceding process naturally suggest themselves, but it is not worth while to go into them.

BALTIMORE, *Sept.* 15, 1896.

ON THE SINGULARITIES OF SINGLE-VALUED AND GENERALLY ANALYTIC FUNCTIONS.

By Prof. Alexandre S. Chessin, Baltimore, Md.

A single-valued function of one variable $f(z)$ will be called *generally analytic* in a domain (D) if it be analytic everywhere in this domain except at points and lines forming a *discrete multiplicity.** It is well known that such functions have no singularities in the domain (D) other than poles or essentially singular points. However, the proof of this fundamental proposition is lacking in rigor in most of the existing treatises on the Theory of Functions, and it may therefore be desirable to supply this deficiency. The weak point of these demonstrations consists in admitting without proof a proposition which is by no means evident, namely: *If at a point a the single-valued and generally analytic function $f(z)$ ceased to be analytic without becoming infinite, then the function $(z - a) f(z)$ would be analytic at this point.*† As a matter of fact all we can affirm is that the function $\varphi(z) = (z - a) f(z)$ is finite and continuous at the point a. But this is only one of the conditions implied in the definition of an analytic function; nothing shows that the condition

$$\frac{\partial \varphi(z)}{\partial y} = i \frac{\partial \varphi(z)}{\partial x} \tag{1}$$

is satisfied at this point *à priori*. We will show in this paper that the condition (1) is really satisfied at the point a and that $f(z)$ cannot cease to be analytic without becoming infinite.

Our proof is based on the following corollary of Green's Theorem:‡

Let $X(x, y)$ and $Y(x, y)$ be two functions of the real variables x and y, which are finite and continuous throughout a connected domain (D), and which generally (i. e. with the exception of points and lines forming a discrete multi-

* We will say that a finite or infinite number of points and lines form a discrete multiplicity if they can be included within areas $\sigma_1, \sigma_2, \sigma_3, \ldots$ such that $\Sigma_i \sigma_i$ can be made arbitrarily small. In this definition we follow the ideas of Hankel. See for example Harnack's Introduction to the study of the Differential and Integral Calculus (English translation by G. Catheart), pp. 243, 296.

† See for example *Forsyth's* Theory of Functions, §§ 32–33, pp. 52 and 53; or *Durège's* Elements of the Theory of Functions (English translation from the 4th German edition), p. 127.

‡ This form of Green's theorem where Rieman's notion of integrability is introduced is due to A. Harnack. See, for example, his Introduction to the Study of the Diff. and Int. Calculus, p. 315.

plicity) *satisfy the equation* $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$; *let further* (x_0, y_0) *and* (x, y) *be any two points within* (D); *then the definite integral*

$$\int_{(x_0, y_0)}^{(x, y)} (X dx + Y dy) \tag{2}$$

will be independent of the path of integration, provided the several paths lie entirely within (D) *and can be brought to coincide with one another by a continuous deformation without crossing any of the boundary lines of* (D).

It is well to remark that the limits of the curvilinear integral (2) may be points at which the equation $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ is not satisfied.

Let us now put $f(z) = u + iv$, and let us consider the definite integral $\int_{z_0}^{z} f(z)\,dz$. This integral is equal to the sum of two curvilinear integrals, namely :

$$\int_{z_0}^{z} f(z)\,dz = \int_{(x_0, y_0)}^{(x, y)} (u dx - v dy) + i \int_{(x_0, y_0)}^{(x, y)} (v dx + u dy).$$

To each one of these curvilinear integrals we may apply the above proposition, and we thus obtain the following :—

Theorem I. Let $f(z)$ *be a single-valued function which is finite and continuous throughout a connected domain* (D) *and generally analytic in the same; let also* z_0 *and* z *be any two points within* (D) *: then the definite integral*

$$\int_{z_0}^{z} f(z)\,dz$$

will be independent of the path of integration, provided the several paths lie entirely within (D) *and can be brought to coincide with one another by a continuous deformation without crossing any of the boundary lines of* (D).

In fact the functions $u(x, y)$ and $v(x, y)$ are finite and continuous throughout the domain (D), and, moreover, they *generally* satisfy the equations $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$; $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$, since $f(z)$ is generally analytic in the domain (D) by hypothesis.

Theorem I is a generalization of Cauchy's theorem, which assumes that $f(z)$ is analytic *throughout* the given domain.

We will now show that a single-valued and generally analytic function $f'(z)$ cannot be finite and continuous throughout a connected domain unless it be analytic without exception in the same. To this end we will prove the following :—

Theorem II. Let the single-valued function $f(z)$ be finite and continuous throughout a connected domain (D) and generally analytic in the same. If then we put

$$F(z) = \int_{z_0}^{z} f'(z)\, dz \tag{3}$$

we shall have $F'(z) = f'(z)$ throughout the domain (D).

In fact let $z + \Delta z$ be any point in the neighborhood of the point z within the domain (D), and let us consider the integral

$$\int_{z}^{z+\Delta z} f'(z)\, dz\ . \tag{4}$$

By Theorem I we may take for the path of integration in (4) the broken line from z to $z + \Delta x$, and thence to $z + \Delta z$. This broken line will always lie within (D) provided Δz be sufficiently small. Hence

$$\int_{z}^{z+\Delta z} f'(z)\, dz = \int_{z}^{z+\Delta x} f'(z)\, dz + \int_{z+\Delta x}^{z+\Delta z} f'(z)\, dz$$

$$= \int_{x}^{x+\Delta x} f'(x+iy)\, dx + i \int_{y}^{y+\Delta y} f'(x+\Delta x+iy)\, dy\ .$$

On the other hand, $f'(z)$ being by hypothesis continuous throughout the given domain, we have by a well known proposition

$$\int_{x}^{x+\Delta x} f'(x+iy)\, dx = \Delta x\, [f'(x+iy) + \varepsilon_1] = \Delta x\, [f'(z) + \varepsilon_1]$$

$$\int_{y}^{y+\Delta y} f'(x+\Delta x+iy)\, dx = \Delta y\, [f'(x+\Delta x+iy) + \varepsilon_2] = \Delta y\, [f'(z) + \varepsilon_3]$$

where ε_1, ε_2, and ε_3 are arbitrarily small quantities vanishing with Δz. Hence

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f'(z)\, dz = \Delta z\ .\ f'(z) + \varepsilon_1 \Delta x + i\varepsilon_3 \Delta y\ ,$$

from which follows

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = f'(z) + \frac{\varepsilon_1 \Delta x + i\varepsilon_2 \Delta y}{\Delta z}.$$

But

$$\left| \frac{\varepsilon_1 \Delta x + i\varepsilon_2 \Delta y}{\Delta z} \right|$$

is an arbitrarily small quantity vanishing with Δz; hence

$$F'(z) = \lim_{\Delta z = 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f'(z). \qquad \text{Q. E. D.}$$

Theorem II may be stated also as follows: *The definite integral of $f'(z)$ considered as function of its upper limit is an analytic function throughout the domain (D).*

COROLLARY. *A function $f'(z)$ which is single-valued and generally analytic in a given domain becomes discontinuous at the points at which it ceases to be analytic.*

In fact, if $f'(z)$ were continuous at a point a at which it ceases to be analytic, let (D) be the neighborhood of this point. By Theorem II the integral function $F(z)$ would be analytic and $F'(z) = f'(z)$ within (D), i. e. in the neighborhood of the point a. But we know that the derivative of an analytic function is itself an analytic function in the same domain. Hence $f'(z)$ would be analytic at the point a, contrary to our assumption.

We are able now to prove that the condition (1) is really satisfied at a point a, if at this point $f'(z)$ ceased to be analytic without becoming infinite. In fact then the function $\varphi(z) = (z-a) f'(z)$ which is single-valued and generally analytic in the same domain as $f'(z)$, being moreover continuous in the neighborhood of the point a, cannot cease to be analytic at this point by the above Corollary. Hence, the equation (1) will be satisfied at the point a. Q. E. D.

This proposition being established it remains only to proceed in the usual way to show that a single-valued and generally analytic function can have no singularities other than poles or essentially singular points. The function $\varphi(z)$ being single-valued and analytic at the point a we can develop it by Taylor's theorem into a series

$$\varphi(z) = (z-a) f'(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots.$$

The function $f'(z)$ is by hypothesis finite; therefore $\underset{z=a}{Lim}\,[(z-a)f'(z)]=0$ and in consequence $c_0=0$. Hence

$$f'(z)=c_1+c_2(z-a)+\ldots,$$

which shows that $f'(z)$ is analytic at the point a contrary to our assumption. We see, therefore, that *a single-valued and generally analytic function $f'(z)$ cannot cease to be analytic without becoming at the same time infinite.*

Now, there are two ways in which a function may become infinite at a point a: Either $\underset{z=a}{Lim}\,f'(z)=\infty$ *uniformly*, i. e. whatever be the path along which the point z tends to the point a; or $\underset{z=a}{Lim}\,f(z)=\infty$ *non-uniformly*, i. e. only as z tends to a along certain paths while along other paths the value of $\underset{z=a}{Lim}\,f'(z)$ is different. Singular points of the first kind are called poles of $f'(z)$; singular points of the second kind are called *essentially singular* points of $f'(z)$. Thus the study of the singularities of single-valued and generally analytic functions is reduced to the study of these two types.

NOTE.—This paper was written and sent to the Annals of Mathematics by the author some time last May, *i. e.* before the appearance of Mr. W. F. Osgood's article on "some points in the elements of the theory of functions" in the Bulletin of the Amer. Math. Society (June, 1896). Mr. Osgood gives two very interesting and simple proofs of the proposition discussed in this paper. The demonstration given here is really a modification of Riemann's proof, and is extracted from the author's lectures on the elements of the theory of functions at the Johns Hopkins University.

ON TRIANGLES WITH RATIONAL SIDES AND HAVING RATIONAL AREAS.

By Mr. H. F. Blichfeldt, Stanford University.

In order to find an expression for the sides of a triangle with rational area we may use the formula :

$$\sqrt{\{s(s-a)(s-b)(s-c)\}} = T,$$

which shows that the condition for rationality is

$$s(s-a)(s-b)(s-c) = T^2,$$

a, b, c and T being rational numbers.

No three of the numbers s, $s-a$, $s-b$ and $s-c$ can have a common factor, if we restrict the sides of the triangle so that they shall have no factor in common. For any factor common to s, $s-a$, and $s-b$, say, would also be common to a, b and c. By this restriction we avoid to deal with triangles that are similar to any one fundamental triangle.

It has been proved, I believe, that $s = \frac{a+b+c}{2}$ is an integer for such triangles. However, it is easy to see that it must be an integer. For let $2s = p$, and by the formula

$$s(s-a)(s-b)(s-c) = T^2$$

we get

$$p(p-2a)(p-2b)(p-2c) = P^2, \text{ say,}$$

where p and P are integers. Hence

$$p^4 - p^3(2a+2b+2c) \equiv P^2 \pmod 4,$$

or

$$p^4 - p^3 \cdot 2p \equiv -p^4 \equiv P^2 \pmod 4,$$

giving

$$p^4 + P^2 \equiv 0 \pmod 4.$$

But this is an impossibility unless p and P are both even. That is, $p = 2s$, where s is an integer.

Any two of the numbers s, $s-a$, $s-b$ and $s-c$ may, however, have a common factor. Let us denote the H. C. F. to s and $s-a$ by α, to s and $s-b$ by β, to $s-c$ and $s-b$ by γ, to $s-c$ and $s-a$ by δ, to s and $s-c$ by ε, and to $s-a$ and $s-b$ by ζ. The factors of the product $s(s-a)(s-b)(s-c) = T^2$ thus assumed are :

$$\alpha^2\beta^2\gamma^2\delta^2\varepsilon^2\zeta^2,$$

a square. The remaining factors must therefore also form a square; they are, moreover, prime to each other, and each must consequently be a square. Let us then denote the remaining factors of s, $s - a$, $s - b$ and $s - c$ by p^2, q^2, r^2, t^2 respectively.

We have now the equations:

$$\left.\begin{aligned} s &= \alpha_1\beta\varepsilon p^2 , \\ s - a &= \alpha\delta\zeta q^2 , \\ s - b &= \beta\gamma\zeta r^2 , \\ s - c &= \gamma\delta\varepsilon t^2 . \end{aligned}\right\} \qquad \text{I}$$

with the condition:

$$-\alpha_1\beta\varepsilon p^2 + \alpha\delta\zeta q^2 + \beta\gamma\zeta r^2 + \gamma\delta\varepsilon t^2 = 0 \qquad \text{II}$$

since:

$$-s + (s - a) + (s - b) + (s - c) = 0 .$$

It may here be noticed, that α is prime to r and t, β to q and t, γ to p and q, etc. For, if α were not prime to r, β would not be the H. C. F. to s and $s - b$; and so for the others.

The equations (I) and (II) will give expressions for the sides of the triangles having rational areas; the sides a, b, c being integers and freed from common factors.

Solving equation (II) for ζ we find:

$$\zeta = \varepsilon \cdot \frac{\alpha_1\beta p^2 - \gamma\delta t^2}{\alpha\delta q^2 + \beta\gamma r^2} .$$

Substituting this value for ζ in equations (1) and multiplying them by $\alpha\delta q^2 + \beta\gamma r^2$ we obtain:

$$\left.\begin{aligned} s &= \alpha_1\beta p^2 (\alpha\delta q^2 + \beta\gamma r^2) \\ s - a &= \alpha\delta q^2 (\alpha_1\beta p^2 - \gamma\delta t^2) \\ s - b &= \beta\gamma r^2 (\alpha_1\beta p^2 - \gamma\delta t^2) \\ s - c &= \gamma\delta t^2 (\alpha\delta q^2 + \beta\gamma r^2) \end{aligned}\right\} . \qquad \text{III}$$

Any three of these equations will give all triangles of the type sought by varying the arbitrary numbers α, β, γ, etc. But we must divide the right-hand members of the equations by the H. C. F. to:

$$\alpha_1\beta p^2 - \gamma\delta t^2 \text{ and } \alpha\delta q^2 + \beta\gamma r^2 .$$

Solving equations (III) for a, b and c we find:

$$\left.\begin{aligned} a &= \alpha\gamma\,(\beta^2p^2r^2 + \delta^2q^2t^2) \\ b &= \beta\delta\,(\alpha^2p^2q^2 + \gamma^2r^2t^2) \\ c &= (\alpha\delta q^2 + \beta\gamma r^2)\,(\alpha\beta p^2 - \gamma\delta t^2) \end{aligned}\right\}. \qquad \text{IV}$$

As the H. C. F. to $\alpha\delta q^2 + \beta\gamma r^2$ and $\alpha\beta p^2 - \gamma\delta t^2$ is common to all of the numbers s, $s - a$, $s - b$ and $s - c$ (see equations (III)), it is also common to a, b and c, and must be divided out from the right-hand members of equations (IV).

The sides a and b can have no common factor unless it be a factor of $\beta^2p^2r^2 + \delta^2q^2t^2$, since a, β, γ and δ are prime to each other, and $\alpha\gamma$ is prime to $\alpha^2p^2q^2 + \gamma^2r^2t^2$. For a similar reason a and c can have no factor in common unless it be a factor of $\beta^2p^2r^2 + \delta^2q^2t^2$. But this is the sum of two squares, prime to each other, and by a well-known theorem any factor of such a number is again the sum of two squares. We have thus the property of such triangles, that *if any two sides have a common factor, that factor must be the sum of two squares.*

The expressions in (IV) for the sides a, b, c may be put in a symmetrical form. Multiplying the right-hand members of the equations by $pqrt$, then putting A for αpq, B for βpr, C for γrt, and D for δqt, and we get the equations:

$$\left.\begin{aligned} a &= AC\,(B^2 + D^2) \\ b &= BD\,(A^2 + C^2) \\ c &= (BC + AD)\,(AB - CD) \end{aligned}\right\}. \qquad \text{V}$$

Multiply now by $p_1^2 + q_1^2$, and let

$$(B^2 + D^2)\,(p_1^2 + q_1^2) = (Bp_1 + Dq_1)^2 + (Dp_1 - Bq_1)^2 = r_1^2 + s_1^2, \text{ say};$$

$$(A^2 + C^2)\,(p_1^2 + q_1^2) = (Ap_1 - Cq_1)^2 + (Cp_1 + Aq_1)^2 = t_1^2 + u_1^2,$$

so that

$$Bp_1 + Dq_1 = r_1,\ Dp_1 - Bq_1 = s_1,\ Ap_1 - Cq_1 = t_1,\ Cp_1 + Aq_1 = u_1.$$

If we now solve these equations for A, B, C and D, substitute in the equations given above (V) and multiply the right-hand members by $(p_1^2 + q_1^2)^3$, we obtain:

$$\left.\begin{aligned} a &= (p_1t_1 + q_1u_1)\,(p_1u_1 - q_1t_1)\,(r_1^2 + s_1^2), \\ b &= (p_1s_1 + q_1r_1)\,(p_1r_1 - q_1s_1)\,(t_1^2 + u_1^2), \\ c &= (r_1u_1 + s_1t_1)\,(r_1t_1 - s_1u_1)\,(p_1^2 + q_1^2). \end{aligned}\right\} \qquad \text{VI}$$

This result could also have been obtained by dividing out a supposed common factor to $B^2 + D^2$ and $A^2 + C^2$ from equations (V), which would also be a factor of $(BC + AD)(AB - CD)$. Let that factor be $p_1^2 + q_1^2$, and by a known proposition we see that we ought to have:

$$B = p_1 r_1 - q_1 s_1,\ D = p_1 s_1 + q_1 r_1,\ A = p_1 t_1 \pm q_1 u_1,\ C = p_1 u_1 \mp q_1 t_1,$$

assuming $B^2 + D^2 = (p_1^2 + q_1^2)(r_1^2 + s_1^2)$, and $A^2 + C^2 = (p_1^2 + q_1^2)(t_1^2 + u_1^2)$. By the substitution of these values for A, B, C and D, using the upper sign in A and C (in order that $p_1^2 + q_1^2$ may divide $(BC + AD)(AB - CD)$), we will get the equations (VI), the right-hand members multiplied by the assumed common factor $p_1^2 + q_1^2$.

By means of equations (V) we can get a very simple expression for the sides of the triangle.

If we divide the numbers expressing the sides by $ABCD$, we have:

$$a = \left[\frac{B}{D} + \frac{D}{B}\right],$$

$$b = \left[\frac{A}{C} + \frac{C}{A}\right],$$

$$c = \left[\frac{B}{D} + \frac{A}{C}\right]\left[1 - \frac{C}{A} \cdot \frac{D}{B}\right].$$

Let us now put $m = \frac{B}{D}$ and $n = \frac{A}{C}$, and the equations become:

$$\left.\begin{aligned} a &= m + \frac{1}{m}, \\ b &= n + \frac{1}{n}, \\ c &= (m+n)\left[1 - \frac{1}{m \cdot n}\right] = m - \frac{1}{m} + n - \frac{1}{n}. \end{aligned}\right\} \quad \text{VII}$$

By giving to m and n all rational values—integral or fractional, positive or negative—we form all the triangles possible with rational sides and having rational areas.

If, by the preceding formulæ, one or more of the sides come out negative, we need merely to change their signs.

ON THE FUNDAMENTAL PROBLEM OF THE DIFFERENTIAL CALCULUS.

By Prof. W. H. Echols, Charlottesville, Va.

1. The fundamental problem of the Differential Calculus may be stated thus: Given the values of a function and its derivatives for a value a of the variable, to determine the values of the function and its derivatives for another value x of the variable. The general solution of this problem results in the Taylor-Lagrange formula for the expansion of the function in infinite series. The presentation of this all important theorem by the text books used in America for undergraduate students is wholly unsatisfactory. One need only refer to the treatise of Todhunter or Williamson in order to substantiate this statement. Either the formula is tentatively laid down to start with or the vicious assumption of the infinite series with undetermined coefficients is made and the series differentiated inexcusably in order to evaluate the coefficients; then subsequently, usually after applications to specific functions, Lagrange's theorem on the remainder is deduced and the so-called "cases of failure" commented upon.

2. The *fundamental principle* of the Calculus is the calculation of the function which is the limit of the finite difference ratio

$$\frac{f(x) - f(a)}{x - a},$$

as x converges to the limit a. This process is called differentiation, and the resulting function called the first derivative of $f(x)$ at a. Repeated application of this process results in the successive derivatives of $f(x)$ at a.

3. The *fundamental problem* of the Calculus is solved by repeated application of the process of differentiation to this same finite difference ratio

$$\frac{f(x) - f(a)}{x - a}.$$

Let $f(x)$ be a one-valued function of x which is finite and continuous throughout an interval (α,β) of the variable. Let x and a be two values of the variable in the interval (α,β). Let the first n derivatives of $f(x)$ be finite and determinate in the neighborhood of a. Apply Leibnitz formula for the differentiation of a product to the successive differentiation of the function

$$\frac{f(x) - f(a)}{x - a}$$

n times with respect to a. Whence, after multiplying both sides by $(x-a)^{n+1}/n!$, then results

$$f(x) - \sum_{0}^{n}{}_{r} \frac{(x-a)^r}{r!} f^r(a) = \frac{(x-a)^{n+1}}{n!} \left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a}. \qquad (1)$$

The member on the right represents the remainder at once in definite form, which can easily be thrown into Lagrange's form in the usual way.

Let z be a value of the variable in the interval (ax), and let

$$\varphi(z) = f(x) - \sum_{0}^{n}{}_{r} \frac{(x-z)^r}{r!} f^r(z) - \frac{(x-z)^{n+1}}{n!} \left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a}.$$

If the first n derivatives of $f(x)$ are finite and continuous throughout (ax) the function $\varphi(z)$ will be also. We have $\varphi(x) = 0$, also $\varphi(a) = 0$ in virtue of (1). Therefore $\varphi'(z) = 0$ for some value u of z between x and a, and we have

$$\left[\frac{d}{da}\right]^n \frac{f(x)-f(a)}{x-a} = \frac{f^{n+1}(u)}{n+1}. \qquad a < u < x.$$

This completes the formula

$$f(x) = \sum_{0}^{n}{}_{r} \frac{(x-a)^r}{r!} f^r(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u). \qquad (2)$$

If now the successive derivatives are determinate at a, and the $(n+1)$th derivative of $f(x)$ for all points between a and x is such that

$$\mathop{\mathrm{L}}_{n=\infty} \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u) = 0,$$

we have

$$f(x) = \sum_{0}^{\infty}{}_{r} \frac{(x-a)^r}{r!} f^r(a), \qquad (3)$$

throughout the interval.

4. The process usually employed for determining Lagrange's theorem on the "remainder" in which it is assumed that the remainder after the $(n+1)$th term can be written, in general, in the form $(x-a)^m Q$ in which Q is assumed to be some function of only x and a is not to be approved on first reading without careful examination, if indeed it is a legitimate assumption under any circumstances. This assumption leads to the result

$$R_{n+1} = \frac{(1-\theta)^{n+1-m}}{m} \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u), \qquad 0 < \theta < 1$$

in which m is a function of u and must not be greater than $n+1$. The method of Art. 3 seems to be free from such objections inasmuch as (1) is an algebraical identity depending only on the process of differentiation of a function of finite form.

5. If (3) be true, then throughout the same interval, we must also have

$$f'(x) = \sum_{1}^{\infty}{}_{r} \frac{(x-a)^{r-1}}{(r-1)!} f^{r}(a) .$$

For, differentiating (1), we have

$$f'(x) = \sum_{1}^{n}{}_{r} \frac{(x-a)^{r-1}}{(r-1)!} f^{r}(a) + \frac{(x-a)^{n}}{n!} \left\{ (n+1) P + (x-a) \frac{dP}{dx} \right\},$$

wherein

$$P = \left[\frac{d}{da} \right]^{n} \frac{f(x) - f(a)}{x-a} = \frac{f^{n+1}(u)}{n+1} .$$

Let

$$J(z) = f'(x) - \sum_{1}^{n}{}_{r} \frac{(x-z)^{r-1}}{(r-1)!} f^{r}(z) - \frac{(x-z)^{n}}{n!} Q ,$$

wherein

$$Q = (n+1) P + (x-a) \frac{dP}{dx} ,$$

and is independent of z.

As before, if the first n derivatives of $f(z)$ are finite and continuous throughout (ax), the function $J(z)$ will be also, for no higher derivatives are involved. We have $J(x) = 0$ and $J(a) = 0$, consequently $J'(v) = 0$ where v is some value of the variable in (ax). Hence

$$Q = f^{n+1}(v) ,$$

and we have

$$f'(x) = \sum_{1}^{n}{}_{r} \frac{(x-a)^{r-1}}{(r-1)!} f^{r}(a) + \frac{(x-a)^{n}}{n!} f^{n+1}(v) ,$$

and since

$$\frac{(x-a)^{n}}{n!} f^{n+1}(v) \text{ and } \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u)$$

vanish when $n = \infty$ under the same circumstances, we have

$$f'(x) = \sum_{1}^{\infty}{}_{r} \frac{(x-a)^{r-1}}{(r-1)!} f^{r}(a) ,$$

for the same values of the variable for which (3) holds. And in general,

$$f^{m}(x) = \sum_{m}^{\infty}{}_{r} \frac{(x-a)^{r-m}}{(r-m)!} f^{r}(a) ,$$

throughout the same interval for which (3) is true.

Thus the general problem is solved, the series (3) giving the value of the function at x in terms of the function and its derivatives at a. The successive derivatives of this series are shown to be the successive derivatives of the function at x.

JUNE, 1896.

PROOF OF A THEOREM IN CONTINUED FRACTIONS.

By Mr. Derrick N. Lehmer, Lincoln, Neb.

The theorem in question is : If $\sqrt{R}$ be expressed as a continued fraction ;

$$\sqrt{R} = q_0 + \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} + \frac{1}{q_1} + \cdots$$

then $q_n = 2q_0$. The proofs usually given of this theorem require so many auxiliary propositions that the following proof may be as welcome to others as it was to me.

Let

$$x = \sqrt{R} - q_0 = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} + \frac{1}{q_1} + \cdots .$$

Denote the nth convergent fraction by A_n/B_n ; then

$$\frac{1}{x} = \frac{A_n/x + A_{n-1}}{B_n/x + B_{n-1}}$$

or

$$x^2 + x(A_n - B_{n-1})/A_{n-1} - B_n/A_{n-1} = 0 .$$

One root of this equation is $\sqrt{R} - q_0$ and since the coefficients are rational the other root must be $-\sqrt{R} - q_0$, and the equation in x is

$$x^2 + 2q_0x - (N - q_0^2) = 0 .$$

Equating coefficients we have $(A_n - B_{n-1})/A_{n-1} = 2q_0$.

Now $A_n = q_n A_{n-1} + A_{n-2}$, whence

$$q_n + (A_{n-2} - B_{n-1})/A_{n-1} = 2q_0 .$$

Unless $(A_{n-2} - B_{n-1})/A_{n-1}$ is an integer the last equation cannot hold, as we would then have an integer equal to a fraction. Since then $A_{n-2} < A_{n-1}$ the fraction must be zero, or $A_{n-2} = B_{n-1}$, which is, I believe, a new proof of the theorem. We have at once then

$$q_n = 2q_0 .$$

October, 1896.

CONTENTS.

Page.

Invariants of Curves and Surfaces of the Second Degree by the Group of Motions and the Group of Similitude. By Dr. E. O. Lovett, 33

Solution of a System of Equations Occurring in Darboux's *Théorie Générale des Surfaces*. By Dr. T. Craig, ... 48

On the Singularities of Single-valued and Generally Analytic Functions. By Prof. Alexandre S. Chessin, ... 52

On Triangles with Rational Sides and having Rational Areas. By Mr. H. F. Blichfeldt, ... 57

On the Fundamental Problem of the Differential Calculus. By Prof. W. H. Echols, ... 61

Proof of a Theorem in Continued Fractions. By Mr. Derrick N. Lehmer, ... 64

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